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ASYMPTOTIC MODULUS RESULTS FOR COMPOSITES CONTAINING RANDOMLY ORIENTED FIBERS

R. M. CHRISTENSEN
Department of Mechanical Engineering and Materials Research Laboratory, Washington University, St. Louis, MO 6313, U.S.A.

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Abstract—Asymptotic methods are used to obtain simple formulas for use in predicting the effective modulus properties of composite materials containing randomly oriented fibers. Both cases of two and three dimensional random orientation are treated. The theoretical predictions are compared with experimental results in the two-dimensional case. The results extend and simplify an earlier approach to the problem.

INTRODUCTION

The fundamental problem in the theory of fiber reinforcement is that of providing predictive methods for the properties. The end results of predictive derivations are the formulas which provide the basis of design methods. In terms of the stiffness properties, the primary work to date has concerned aligned fiber systems. In fact, references to the aligned fiber case are too numerous to detail here. Suffice to say, great success has been derived from these theoretical results, and design methods abound for aligned fiber system. Recent developments in this field have focused attention upon the complementary configuration involving random fiber orientation. The random fiber case admits description in both two and three dimensions, corresponding to planar, and fully three dimensional cases. The two dimensional, plane stress case represents the configuration of a thin mat of randomly oriented fibers impregnated by a resin phase. The three dimensional case is becoming of technological importance because of the emergence of a new processing method based upon pellets which are pre-compounded fiber-resin combinations. The use of these percompounded pellets in conventional processing methods results in random orientations of the fibers, or partially randomized configurations.

There have been several previous studies of the random fiber case. The first such study was apparently that due to Cox[1]. The motivation of this work was to obtain a predictive method for the properties of cellulose fiber materials. The result was the Cox formula for the three dimensionally random case

$$E_{3D} = \frac{cE_f}{6}$$

where E_f is the modulus of the fiber phase, and c its volume concentration. The corresponding result for the two dimensional case was found to be

$$E_{2D} = \frac{cE_f}{3}.$$

The Cox formulii do not account for the presence of a matrix phase and the resulting interaction between fiber and matrix phases. The first account of fiber-matrix interaction effects was provided by Tsai and Pagano[2] in the context of laminated plates. Further work along the lines using the "laminate analogy" was given by Halpin, Jerina and Whitney[3]. A numerical method for predicting the properties was given by Nielsen and Chen[4]. Finally, Christensen and Waals[5] provided a method for determining random orientation properties using a geometric averaging method.

All of the above noted methods concerning random fiber-matrix systems require some type of algebraic/numerical evaluation as the final step. That is to say, these methods do not result in final analytical forms which directly admit physical interpretation. The somewhat complicated final forms of these results inhibits their widespread application and utilization. There is a need for simple, reliable formulas which predict the random orientation composite properties in terms of the properties of each phase. Formulas corresponding to the rule of mixtures for the aligned fiber case are needed for the random orientation case. However, these simple formulas, suitable for design application, cannot be conjectured or postulated, and preferably they should not be empirical forms established from limited data examination. Ideally, these predictive property forms should be rationally derived from the principles of mechanics. This is the aim of the present study. If successful, the resulting expressions will provide forms that invite physical interpretations and understanding, as well as ease of application.

The plan here is to recall the appropriate formulation for Christensen and Waals [5]. However, a method will be introduced by which the results of Ref. [5] can be refined to a much higher degree than was accomplished previously. The key to this new method depends upon the assumption that the fiber phase is much stiffer than the matrix phase. With this assumption, a certain small parameter is introduced and an asymptotic expansion is used to obtain the desired result. Of course the results of this method are more restrictive that those of Ref. [5], and it must be proved that the results possess validity for the practical systems of interest; this will be done.

GENERAL FORMULATION

Several results will now be stated which are needed in the further derivation. First, the five independent effective properties of an aligned fiber system are given by

$$E_{11} = cE_f + (1-c)E_m + 4c(1-c)\mu_m\left[\frac{(\nu_f-\nu_m)^2}{\dfrac{(1-c)\mu_m}{k_f+\mu_m/3}+\dfrac{c\mu_m}{k_m+\mu_m/3}+1}\right] \tag{1}$$

$$\nu_1 = \nu_{12} = \nu_{13} = c\nu_f + (1-c)\nu_m + \frac{c(1-c)(\nu_f-\nu_m)\left(\dfrac{\mu_m}{k_m+\mu_m/3}-\dfrac{\mu_m}{k_f+\mu_f/3}\right)}{\dfrac{(1-c)\mu_m}{k_f+\mu_f/3}+\dfrac{c\mu_m}{k_m+\mu_m/3}+1} \tag{2}$$

$$K_{23} = k_m + \frac{\mu_m}{3} + \frac{c}{\dfrac{1}{k_f-k_m+(\mu_f-\mu_m)/3}+\dfrac{(1-c)}{k_m+(4\mu_m/3)}} \tag{3}$$

$$\frac{\mu_{12}}{\mu_m} = \frac{\mu_f(1+c)+\mu_m(1-c)}{\mu_f(1-c)+\mu_m(1+c)} \tag{4}$$

and finally

$$\frac{\mu_{23}}{\mu_m} = 1 + \frac{c}{\dfrac{\mu_m}{\mu_f-\mu_m}+\dfrac{[k_m+7\mu_m/3](1-c)}{2(k_m+4\mu_m/3)}} \tag{5}$$

where rectangular cartesian coordinates are employed with axis 1 being in the fiber direction. The above results are found from the composite cylinders assemblage of Hashin and Rosen, as discussed in Ref. [5]. Parameter c represents the volume fraction of the fiber phase, with fiber and matrix properties denoted respectively by subscripts f and m. Actually relation (5) is a lower bound upon the corresponding property, μ_{23}, and this will be used in the absence of an exact solution for this particular property.†

In Ref. [5] a method was developed to obtain the isotropic effective properties of a system having randomly oriented fibers. The method involved a geometric averaging procedure whereby aligned fiber systems assume all possible orientations. Although this was not explicitly shown in Ref. [5], the method developed therein can be used to obtain the following results,

†An exact solution for this property has been found recently and will be published. It can be demonstrated that for present purposes, formula (5) gives a sufficiently accurate representation of the exact results.

$$\mu_{3D} = \frac{1}{15}[E_{11} + (1-2\nu_1)^2 K_{23} + 6(\mu_{12} + \mu_{23})] \tag{6}$$

and

$$k_{3D} = \frac{1}{9}[E_{11} + 4(1+\nu_1)^2 K_{23}] \tag{7}$$

when μ_{3D} and k_{3D} are the corresponding effective shear and bulk modulii for systems containing a three dimensionally random fiber orientation. Substitution from relations (1)–(5) into (6) and (7) permits the determination of the composite properties in terms of the fiber and matrix phase properties. Unfortunately the final form of relations (6) and (7), after substitution from (1)–(5), is very complicated, and it does not allow simple, direct interpretation. It is the purpose of this paper to rationally deduce simplified property forms from relations (6) and (7), under certain conditions. The primary conditions to be used is that the fiber phase is much stiffer than the matrix, an entirely realistic condition for all practical systems. First the three dimensional case will be considered after which the two dimensional, plane stress case will be studied.

THREE DIMENSIONAL RANDOM FIBER ORIENTATION

As mentioned above, it will be assumed that the fibers are very stiff relative to the matrix properties. In the initial procedure to find simplified modulus expressions, the dilute suspension case will be invoked. That is, the fiber phase will be taken to be in a sufficiently low volume concentration such that only terms of order zero and one in c need be retained in the modulii expressions. Actually this approach will be shown to yield only very limited results, but the resulting deficiencies are illuminating as regards the search for a more viable approach.

Assume $c \ll 1$, $E_f \gg E_m$ and $E_f \gg k_m$. Under these assumptions it is obvious from (1)–(7) that

$$\left.\begin{aligned} \mu_{3D} &= \frac{c}{15}E_f + \mu_m \\ k_{3D} &= \frac{c}{9}E_f + k_m \end{aligned}\right\} \begin{aligned} &c \ll 1 \\ &E_f \gg E_m \\ &E_f \gg k_m. \end{aligned} \tag{8}$$

Actually it is not μ_{3D} and k_{3D} which are of primary engineering interest, rather it is the elasticity modulus, E_{3D}, which should be obtained. This is of course determined by the usual isotropic relation

$$E_{3D} = \frac{9k_{3D}\mu_{3D}}{3k_{3D} + \mu_{3D}}. \tag{9}$$

However, in substituting from relations (8) into (9) three separate cases arise. These are the cases $cE_f \ll E_m$, $cE_f \equiv E_m$, and $cE_f \gg E_m$. In the second of these cases no simplification is possible after substituting from (8) into (9). In the first case, $cE_f \ll E_m$, (9) can be written in the form of a power series in (cE_f/E_m). Retaining up to first order results gives, after algebraic reduction

$$E_{3D} = E_m + \left(\frac{3 - 4\nu_m + 8\nu_m^2}{15}\right)cE_f \quad \left\} \begin{aligned} &cE_f \ll E_m \\ &c \ll 1 \\ &E_f \gg E_m \\ &E_f \gg k_m. \end{aligned}\right. \tag{10}$$

Alternatively, in the case, $cE_f \gg E_m$, relations (8) and (9) can be combined into the form of a power series in (E_m/cE_f). The first two terms of this expansion are given by

$$E_{3D} = \frac{c}{6}E_f + \frac{(9-16\nu_m)}{8(1+\nu_m)(1-2\nu_m)}E_m \quad \left\} \begin{aligned} &cE_f \gg E_m \\ &c \ll 1 \\ &E_f \gg E_m \\ &E_f \gg k_m. \end{aligned}\right. \tag{11}$$

While relations (10) and (11) are valid under the restrictions used in their derivation, they are of limited usefulness, because of the restriction to $c \ll 1$, thus only the slope at the origin of the E_{3D} vs c curve is obtained. However, one very important observation emerges from the preceding derivation. Note the necessity to choose between the conditions $cE_f \ll E_m$ or $cE_f \gg E_m$. For most practical purposes the latter condition, $cE_f \gg E_m$, is realistic. This observation suggests that it may be possible to obtain asymptotic results using a power series expansion in powers of (E_m/cE_f) rather than in powers of c as was done in the preceding derivation. In fact, it will be shown that this is possible, a power series expansion in (E_m/cE_f) will be obtained with no restriction that c itself be small.

Proceeding according to the above lines of reasoning, substitute directly from (6) and (7) into (9). There results

$$E_{3D} = \frac{cE_f + [2\hat{E}_{11} + (5+4\nu_1+8\nu_1^2)K_{23} + 6(\mu_{12}+\mu_{23})] + 0(1/cE_f)}{6\left[1 + \dfrac{2\hat{E}_{11} + (7+12\nu_1+8\nu_1^2)K_{23} + 2(\mu_{12}+\mu_{23})}{2cE_f}\right]} \tag{11}$$

where

$$\hat{E}_{11} + E_{11} - cE_f \tag{12}$$

with $cE_f \gg E_m$ and $cE_f \gg k_m$. The quotient in (11) can be obtained as a power series in (E_m/cE_f). The first two terms of this power series are given by

$$\frac{E_{3D}}{E_m} = \frac{1}{6}\frac{cE_f}{E_m} + \frac{\hat{E}_{11}}{6E_m} + \frac{(3-4\nu_1+8\nu_1^2)K_{23}}{12E_m} + \frac{5}{6}\frac{(\mu_{12}+\mu_{23})}{E_m} + 0\left(\frac{E_m}{cE_f}\right). \tag{13}$$

Now, under the conditions of $cE_f \gg E_m$ and $cE_f \gg k_m$, relations (1)–(5) along with (12) can be reduced to the forms

$$\hat{E}_{11} = (1-c)E_m + 4c(1-c)\mu_m\left[\frac{(\nu_f-\nu_m)^2}{\dfrac{c\mu_m}{k_m+\mu_m/3}+1}\right] \tag{14}$$

$$\nu_1 = c\nu_f + (1-c)\nu_m + \frac{c(1-c)(\nu_f-\nu_m)\left(\dfrac{\mu_m}{k_m+\mu_m/3}\right)}{1+\dfrac{c\mu_m}{k_m+\mu_m/3}} \tag{15}$$

$$K_{23} = \frac{k_m}{1-c} + \frac{(1+3c)}{(1-c)}\frac{\mu_m}{3} \tag{16}$$

$$\mu_{12} = \frac{(1+c)}{(1-c)}\mu_m \tag{17}$$

and

$$\mu_{23} = \mu_m + \frac{2c\left(k_m + \dfrac{4}{3}\mu_m\right)}{(1-c)\left(k_m + \dfrac{7}{3}\mu_m\right)}\mu_m \tag{18}$$

although it must be noted that these formulii are not valid at $c = 1$.

The substitution of relations (14)–(18) into (13) completes the derivation. However, the results are still somewhat complicated unless specific values of Poisson's ratio are assigned. The case of $\nu_f = \nu_m = \frac{1}{4}$ has been computed, with the result from (13)–(18) that

$$\left.\begin{aligned}\hat{E}_{11} &= (1-c)E_m\\ \nu_1 &= \frac{1}{4}\\ K_{23} &= \frac{2}{5}\frac{(2+c)}{(1-c)}E_m\\ \mu_{12} &= \frac{2}{5}\frac{(1+c)}{(1-c)}E_m\\ \mu_{23} &= \frac{1}{5}\frac{(2+c)}{(1-c)}E_m\end{aligned}\right\}\quad \begin{aligned}&\nu_f = \nu_m = \frac{1}{4}\\ &E_f \gg E_m\\ &E_f \gg k_m\\ &c \neq 1.\end{aligned} \tag{19}$$

Substitution of (19) into (13) gives

$$\left.\frac{E_{3D}}{E_m}\right|_{\nu_f=\nu_m=1/4} = \frac{1}{6}\frac{cE_f}{E_m} + \left(\frac{1+c/4+c^2/6}{1-c}\right) + 0\left(\frac{E_m}{cE_f}\right). \tag{20}$$

The practical usefulness of this formula will be assessed later. First a completely different means of solving for E_{3D} will be considered.

Under conditions that both phases are incompressible, the second term in (6) can be shown to vanish leaving

$$\mu_{3D}|_{\nu_f=\nu_m=1/2} = \frac{1}{15}[E_{11} + 6(\mu_{12} + \mu_{23})]. \tag{21}$$

Substituting for E_{11}, μ_{12} and μ_{23} from (1), (4) and (5), with the incompressibility condition, into (21) gives

$$\mu_{3D}|_{\nu_f=\nu_m=1/2} = \frac{c}{5}\mu_f + \frac{1}{5}\left[\frac{(5+2c+c^2)\mu_f + (5+c)(1-c)\mu_m}{(1-c)\mu_f + (1+c)\mu_m}\right]\mu_m. \tag{22}$$

With incompressibility, $E_{3D} = 3\mu_{3D}$ and from (22) there results

$$E_{3D}|_{\nu_f=\nu_m=1/2} = \frac{c}{5}E_f + \frac{1}{5}\left[\frac{(5+2c+c^2)E_f + (5+c)(1-c)E_m}{(1-c)E_f + (1+c)E_m}\right]E_m. \tag{23}$$

Note that relation (23) is an exact result, obtained from relations (1)–(5) whereas the form (20) was based upon an asymptotic expansion method. Relations (20) and (23) certainly are significantly different, and their comparison and usefulness will be determined after first considering the two dimensional case.

TWO DIMENSIONAL, PLANE STRESS, RANDOM FIBER ORIENTATION

Now consider the case of fibers constrained to be in a plane with otherwise random orientation. Under plane stress conditions it was shown in Ref. [5] that the two dimensional modulus is given by

$$E_{2-D} = \frac{1}{u_1}(u_1^2 - u_2^2) \tag{24}$$

where

$$\begin{aligned}u_1 &= \frac{3}{8}cE_f + \hat{u}_1\\ u_2 &= \frac{1}{8}cE_f + \hat{u}_2\end{aligned} \tag{25}$$

with

$$\hat{u}_1 = \frac{3}{8}\hat{E}_{11} + \frac{\mu_{12}}{2} + \frac{(3+2\nu_1+3\nu_1^2)\mu_{23}K_{23}}{2(\mu_{23}+K_{23})} \tag{26}$$

$$\hat{u}_2 = \frac{1}{8}\hat{E}_{11} - \frac{\mu_{12}}{2} + \frac{(1+6\nu_1+\nu_1^2)\mu_{23}K_{23}}{2(\mu_{23}+K_{23})} \tag{27}$$

and where the properties in (26) and (27) are given by (1)–(5) along with (12).

Substituting (25) into (24), the result can be written as

$$\frac{E_{2D}}{E_m} = \frac{cE_f/E_m + 2(3\hat{u}_1 - \hat{u}_2)/E_m + 0(E_m/cE_f)}{3 + 8\hat{u}_1/cE_f}. \tag{28}$$

Writing (28) as a power series expansion in (E_m/cE_f) gives

$$\frac{E_{2D}}{E_m} = \frac{1}{3}\frac{cE_f}{E_m} + \frac{2}{3}\frac{\left(\frac{5}{3}\hat{u}_1 - \hat{u}_2\right)}{E_m} + 0\left(\frac{E_m}{cE_f}\right). \tag{29}$$

Substituting for $\hat{u}_1$ and $\hat{u}_2$ from (26) and (27) into (29) gives

$$\frac{E_{2D}}{E_m} = \frac{1}{3}\frac{cE_f}{E_m} + \frac{1}{E_m}\left[\frac{\hat{E}_{11}}{3} + \frac{8}{9}\mu_{12} + \frac{4}{9}\frac{(3-2\nu_1+3\nu_1^2)}{(\mu_{12}+K_{23})}\mu_{23}K_{23}\right] + 0\left(\frac{E_m}{cE_f}\right). \tag{30}$$

Substitution from (1)–(5) and (12) into (30) completes the formulation. The resulting form however is still quite complicated and it will be evaluated here for specific values of Poisson's ratios. First the case of incompressibility will be considered. for this case

$$\begin{aligned} \nu_1 &= \frac{1}{2} \\ K_{23} &\to \infty \\ E_{11} &= (1-c)E_m \end{aligned} \tag{31}$$

and it can be shown that

$$\mu_{23} = \mu_{12} \tag{32}$$

where μ_{12} is given by (4).

Combining (31), (32) and (4) with (30) gives

$$\left.\frac{E_{2d}}{E_m}\right|_{\nu_f=\nu_m=1/2} = \frac{1}{3}\frac{cE_f}{E_m} + \frac{(1-c)}{3} + \frac{19}{27}\left[\frac{E_f(1+c)+E_m(1-c)}{E_f(1-c)+E_m(1+c)}\right] + 0\left(\frac{E_m}{cE_f}\right). \tag{33}$$

Now the corresponding result will be found for $\nu_f = \nu_m = \frac{1}{4}$. For these values of Poisson's ratio the appropriate stiff fiber case forms are given by relations (19). Combining (19) with (30) provides the final result

$$\left.\frac{E_{2D}}{E_m}\right|_{\nu_f=\nu_m=1/4} = \frac{1}{3}\frac{cE_f}{E_m} + \left[\frac{272-41c+90c^2}{270(1-c)}\right] + 0\left(\frac{E_m}{cE_f}\right). \tag{34}$$

The difference of the zero order term in (E_m/cE_f) in (33) and (34) should be noted. In arriving at (33) it was not necessary to explicitly write μ_{12} and μ_{23} in the form appropriate to stiff fibers, thus both E_f and E_m appear in the zero order term in (33). However in arriving at (34) the formulas (19) are those appropriate to stiff fibers, $E_f \gg E_m$ and $E_f \gg k_m$, thus E_f does not appear in the zero order term in (34).

Neither formula (33) nor (34) is valid at $c = 0$ or $c = 1$. At $c = 0$ the expansion procedure in powers of (E_m/cE_f) is invalidated. The expansion also is invalid at $c = 1$ since at $c = 1$ other terms besides E_{11} contribute a term of the type cE_f, the expansion term; consider for example formula (4) at $c = 1$. Despite the restrictions $c \neq 0$, $c \neq 1$, formulas (33) and (34) will be shown to provide very satisfactory results.

EVALUATION OF RESULTS

First consider the three dimensional random orientation results. Relations (20) and (23) are the final forms, appropriate to different values of Poisson's ratios. In the case where the matrix phase vanishes the results from these forms are as follows

$$E_{3D}\Big|_{\substack{\nu_f=\nu_m=1/4\\ E_m=0}} = \frac{c}{6}E_f \tag{35}$$

and

$$E_{3D}\Big|_{\substack{\nu_f=\nu_m=1/2\\ E_m=0}} = \frac{c}{5}E_f. \tag{36}$$

The difference in these relations is reconciled when one recalls the limit processes involved in reaching the result (36). First there was the incompressibility process $\nu_m \to \frac{1}{2}$ followed by the vanishing modulus process $E_m \to 0$. An examination of the procedure involved reveals that if the order of the processes were reversed, then the form (35) results. In light of these remarks the proper result for the vanishing matrix case is that of relation (35).

Reliable experimental data does not appear to be available in the three dimensional random case. The utility of the formulas (20) and (23) will be studied through comparison with results presented in Ref. [5] for a glass-epoxy system. The comparison is shown in Fig. 1. The expansion formula (20) for $\nu_f = \nu_m = \frac{1}{4}$ is indistinguishable from the Ref. [5] result up to a volume fraction of $c = \frac{1}{2}$. At larger values of the volume fraction, formula (20) increasing deviates from the previous solution. The result (23) for the incompressible case, as applied to the glass-epoxy system, in Fig. 1, exhibits a considerable deviation from the Ref. [5] result which properly incorporates the proper values of Poisson's ratios for the two phases. This result reveals the sensitivity of the three dimensionally random system to variations in Poisson's ratio.

In the two dimensional case, there is reliable experimental data for comparison, see Fig. 2. Relations (33) and (34) are the expansion formulas for the respective cases of $\nu_f = \nu_m = \frac{1}{2}$, $\nu_f = \nu_m = \frac{1}{4}$. Noting the use of the logarithmic scale in Fig. 2, it is seen that the comparison between the theory and the experimental results are very close, and in fact the results of the two

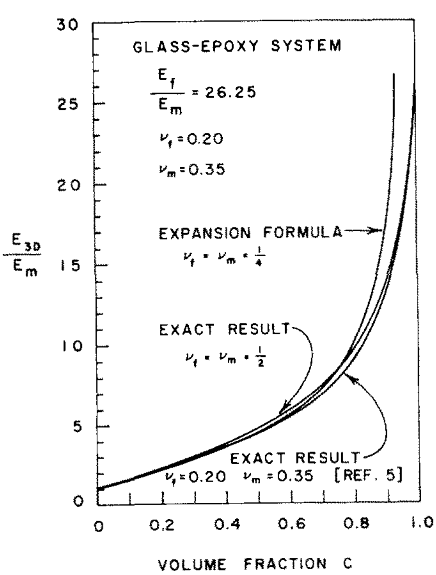

Fig. 1. Three-dimensional random orientation case.

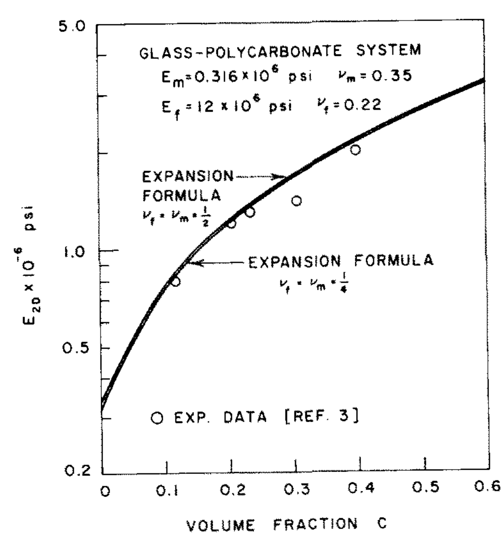

Fig. 2. Two-dimensional random orientation case, comparison with experiment.

formulas are indistinguishable to within experimental error. As one might expect the two-dimensional plane stress case is much less sensitive to variations in Poisson's ratio than is the three dimensional case, as noted in the foregoing discussion.

In view of these results reasonable predictions of the Young's modulus property for stiff, randomly oriented fibers in a compliant matrix phase are afforded by formulas (20) in the three dimensional case and formula (33) in the two dimensional case. Rewriting formulas (20) and (33) here for convenience gives

$$E_{3D} = \frac{c}{6}E_f + \frac{1}{1-c}\left(1 + \frac{c}{4} + \frac{c^2}{6}\right)E_m \quad (c \le 0.5) \tag{37}$$

and

$$E_{2D} = \frac{c}{3}E_f + \frac{(1-c)}{3}E_m + \frac{19}{27}\left[\frac{E_f(1+c)+E_m(1-c)}{E_f(1-c)+E_m(1+c)}\right]E_m \quad (c \neq 0, c \neq 1). \tag{38}$$

The restriction appended to formula (37) follows from the discussion of the results in Fig. 1. The restriction with formula (38) follows from the nature of the expansion process used to obtain (33). The truncation of (33) at the level of terms shown in (38) assumes that $E_m/cE_f \ll 1$. No matter how stiff the fibers may be in comparison to the matrix, this inequality would not be satisfied at $c = 0$. The restriction that $c \neq 1$ in (38) follows from similar although somewhat more complicated lines of reasoning, as discussed in the derivation. A modification of the coefficient of 19/27 in (38) to the value of 18/27 would remove the restriction that $c \neq 0$, $c \neq 1$. This is not done here since it is the form (38) which has a rigorous basis of derivation, furthermore such a modification would change the prediction of formula (38) in its range of derived validity $(0 < c < 1)$.

The formules (37) and (38) provide simple, easy to use predictions of composite properties. However, for cases outside their range of validity recourse should be made to the exact predictive method of Ref. [5]. With regard to the geometric averaging method given in Ref. [5], it represents a method based upon prescribed strains, thus the resulting modulus predictions represent upper bounds.† Accordingly the present results should be viewed as upper bounds. In the case of laminated systems of aligned lamina, the prescription of uniform strains among the lamina corresponds to the requirement of displacement compatibility. Thus the method of Ref. [5] leads to exact results for the special case of laminated plates, and eqn (38) may be viewed accordingly. For all other random orientation systems, the present results correspond to the upper bound results from Ref. [5].

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†An exact solution for this property has been found recently and will be published. It can be demonstrated that for present purposes, formula (5) gives a sufficiently accurate representation of the exact result.